EPFL– Fall 2025
**Series 4**

# Differential Geometry I: Curves and surfaces

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**4.1** If $a$ and $b$ are vectors in $\mathbb{R}^3$ with $a \neq 0$, we say that a vector $c \in \mathbb{R}^3$ is the normal component of $b \in \mathbb{R}^3$ relative to $a \in \mathbb{R}^3$ if $c \perp a$ and there exists $\lambda \in \mathbb{R}$ such that $b = \lambda a + c$. Show that this normal component can be written as

$$c = \frac{(a \times b) \times a}{\|a\|^2}.$$

If $\gamma : I \to \mathbb{R}^n$, prove that its curvature (i.e. the norm of the curvature vector) $\gamma : I \to \mathbb{R}^3$:

$$\kappa_\gamma(u) = \frac{\|\dot\gamma(u) \times \ddot\gamma(u)\|}{V_\gamma^3}.$$

**Solution.** Define $c$ by

$$c = \frac{(a \times b) \times a}{\|a\|^2}.$$

By the well-known properties of the cross product $c \perp a$. It remains to show that $c - b$ is a scalar multiple of $a$.

Using the first Grassmann identity (vector triple product),

$$(a \times b) \times a = \langle a, a \rangle b - \langle a, b \rangle a.$$

Thus

$$c = \frac{\langle a, a \rangle b}{\|a\|^2} - \frac{\langle a, b \rangle a}{\|a\|^2} = b - \lambda a,$$

with $\lambda = \dfrac{\langle a, b \rangle}{\|a\|^2}$, as required.

For a curve $\gamma : I \to \mathbb{R}^n$, if $V_\gamma = \|\dot\gamma\|$ is its speed, we have $\dot\gamma = V_\gamma T$ (where $T = T_\gamma$ is the unit tangent vector. Moreover, from the acceleration decomposition formula, we have

$$\ddot\gamma = V_\gamma^2 K + \dot V_\gamma T.$$

Then

$$\dot\gamma \times \ddot\gamma = (V_\gamma T) \times (V_\gamma^2 K + \dot V_\gamma T) = V_\gamma^3 (T \times K).$$

Since $T \perp K$ and $\|T\| = 1$, we have $\|T \times K\| = \|T\| \, \|K\| = \kappa_\gamma$. Therefore

$$\|\dot\gamma \times \ddot\gamma\| = V_\gamma^3 \kappa_\gamma,$$

which implies the desired expression.

**4.2** Let $\alpha : I \to \mathbb{R}^n$ be a biregular curve, with curvature vector $K_\alpha$ and principal normal $N_\alpha$. Prove that $K_\alpha = \kappa_\alpha N_\alpha$, where $\kappa_\alpha$ is the curvature of $\alpha$.

**Solution.** Using the acceleration decomposition

$$\ddot{\alpha} = \dot{V}\,T + V^2 K,$$

we get

$$\langle \ddot{\alpha}, T \rangle = \dot{V}\langle T, T\rangle + V^2\langle K, T\rangle = \dot{V},$$

since $\langle T, T\rangle = 1$ and $\langle K, T\rangle = 0$. Therefore

$$\ddot{\alpha} - \langle \ddot{\alpha}, T\rangle T = V^2 K.$$

Hence

$$N_\alpha = \frac{V^2 K}{\|V^2 K\|} = \frac{K}{\|K\|} = \frac{K}{\kappa_\alpha},$$

which yields $K_\alpha = \kappa_\alpha N_\alpha$.

**4.3** Prove that the curve $\gamma(t) = (\cosh(t), \sinh(t), t)$ is biregular, then compute its curvature vector and its curvature (the curvature is the norm of the curvature vector).

**Solution.** Compute

$$\dot{\gamma}(t) = (\sinh t, \cosh t, 1), \qquad \ddot{\gamma}(t) = (\cosh t, \sinh t, 0).$$

These two vectors are linearly independent for all $t$, so $\gamma$ is biregular. The speed is

$$V_\gamma(t) = \|\dot{\gamma}(t)\| = \sqrt{(\sinh t)^2 + (\cosh t)^2 + 1} = \sqrt{2}\cosh t.$$

Thus the unit tangent is

$$T_\gamma(t) = \frac{1}{\sqrt{2}}\Big(\tanh t,\ 1,\ \frac{1}{\cosh t}\Big).$$

Differentiate $T_\gamma$:

$$\dot{T}_\gamma(t) = \frac{1}{\sqrt{2}}\Big(\frac{1}{\cosh^2 t},\ 0,\ -\frac{\sinh t}{\cosh^2 t}\Big).$$

So the curvature vector is

$$K_\gamma(t) = \frac{1}{V_\gamma(t)}\dot{T}_\gamma(t) = \frac{1}{2\cosh^3 t}\,(1, 0, -\sinh t),$$

and the curvature (norm) equals

$$\kappa_\gamma(t) = \|K_\gamma(t)\| = \frac{1}{2\cosh^2 t}.$$

**4.4** The *evolute* of a biregular curve $\alpha : I \to \mathbb{R}^n$ is the curve $\beta : I \to \mathbb{R}^n$ defined by

$$\beta(u) = \alpha(u) + \frac{1}{\kappa_\alpha(u)} N_{\alpha,u},$$

where $\rho_\alpha(u) = \frac{1}{\kappa_\alpha(u)}$ is the radius of curvature and $N_{\alpha,u}$ is the principal normal vector. The evolute of a curve is thus the locus of its centers of curvature (= centers of osculating circles).

Compute the evolutes of the following curves:

**(a)** A circle in $\mathbb{R}^n$.

**(b)** A straight line in $\mathbb{R}^n$.

**(c)** The circular helix $\alpha(u) = (a\cos(u), a\sin(u), bu)$ in $\mathbb{R}^3$.

**(d)** The cycloid $\gamma(t) = (r(t - \sin t), r(1 - \cos t))$ in $\mathbb{R}^2$.

Prove that the evolute of the helix is again a helix, and that the evolute of the cycloid is again a cycloid.

**Solution.**

(a) The evolute of a circle degenerates to its center (a single point).

(b) A line is not biregular (its curvature is zero), so the evolute is not defined (trick question).

(c) For the helix $\alpha(u) = (a\cos u, a\sin u, bu)$ compute

$$\dot\alpha(u) = (-a\sin u, a\cos u, b), \qquad \ddot\alpha(u) = (-a\cos u, -a\sin u, 0).$$

The speed is constant $c = \sqrt{a^2 + b^2}$, so

$$T(u) = \frac{1}{c}(-a\sin u, a\cos u, b), \qquad K(u) = \frac{1}{c}\dot T(u) = -\frac{a}{c^2}(\cos u, \sin u, 0).$$

Thus $\kappa(u) = a/c^2$ and the principal normal is $N(u) = K(u)/\kappa = -(\cos u, \sin u, 0)$. The evolute is

$$\beta(u) = \alpha(u) + \frac{1}{\kappa(u)} N(u) = \Big(-\frac{b^2}{a}\cos u,\ -\frac{b^2}{a}\sin u,\ bu\Big),$$

which is again a helix with the same axis (translated and rescaled in radius).

(d) For the cycloid $\gamma(t) = (r(t - \sin t), r(1 - \cos t))$ compute

$$\dot\gamma(t) = (r(1 - \cos t), r\sin t), \qquad \ddot\gamma(t) = (r\sin t, r\cos t).$$

The speed is $V_\gamma(t) = r\sqrt{2(1 - \cos t)}$. One convenient expression for the unit tangent (on an interval where $\sin t \geqslant 0$) is

$$T(t) = \frac{1}{\sqrt{2}}\big(\sqrt{1 - \cos t},\ \sqrt{1 + \cos t}\big).$$

Differentiate $T(t)$ to get (after simplification)

$$\dot{T}(t) = \frac{1}{2\sqrt{2}}\Big(\frac{\sin t}{\sqrt{1-\cos t}},\ -\frac{\sin t}{\sqrt{1+\cos t}}\Big).$$

Then the curvature vector is

$$K(t) = \frac{1}{V_\gamma(t)}\dot{T}(t) = \frac{1}{4r(1-\cos t)}(\sin t, \cos t - 1).$$

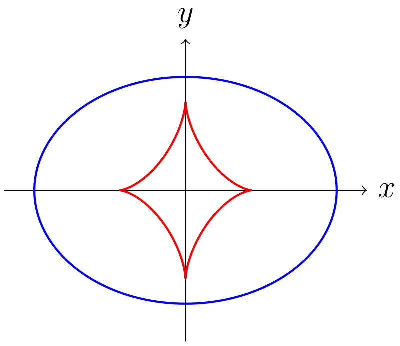
One computes $\kappa(t) = \|K(t)\|$ and finally the evolute

$$\tilde{\gamma}(t) = \gamma(t) + \frac{1}{\kappa(t)}K(t) = (r(t+\sin t),\ r(\cos t - 1)),$$

which is again a cycloid (translated).

**4.5** Without any computation, draw (approximately) an ellipse and its evolute. Explain your reasoning.

**Solution.** By symmetry one may restrict to a quarter of the ellipse. The points where the ellipse intersects its axes are points of minimal and maximal curvature; these correspond to cusps (points of return) on the evolute. Thus the evolute of an ellipse has four cusps, symmetric with respect to the axes.



**4.6** In this exercise, we will study a curve that has been the cornerstone of the design of highways and rail tracks, the so-called *clothoid curve* or *Euler's spiral*.

When driving at a constant speed on a highway (the car can be thought of as tracing out a curve in the plane), the position of the steering wheel determines the rate of "turning" of the velocity vector (and, hence, is related to the acceleration of the car's curve). The rate of turning of the wheel is then associated to the so called "jerk" of the curve (in a natural parametrization, this is the derivative in the curvature of the curve). When designing the portion of a highway that connects two given roads, one aims to minimize abrupt changes of the wheel, distributing the jerk as uniformly as possible along that portion. Thus, such a connecting curve has constant jerk; this curve is called the *clothoid curve*.

**(a)** Derive the differential equation satisfied by the Cartesian expression $\gamma(s) = (x(s), y(s))$ of the clothoid curve parametrized by unit speed.

**(*b)** Obtain an expression for $(x(s), y(s))$ with the initial conditions $\gamma(0) = (0, 0)$, $\dot{\gamma}(0) = (1, 0)$ and $\kappa_\gamma(0) = 0$ in terms of the special functions $\int \cos(s^2)\, ds$, $\int \sin(s^2)\, ds$ *(Hint: Instead of working with the differential equation in Cartesian coordinates, you might want to derive a differential equation for the angle $\theta(s)$ formed by the velocity vector and the x-axis and solve for that first.*

**Solution.**

(a) Since we assumed that the curve $\gamma(s) = (x(s), y(s))$ is parametrized by unit speed, its unit tangent is given by $T_\gamma(s) = \dot{\gamma}(s) = (\dot{x}(s), \dot{y}(s))$ and its curvature vector by $K_\gamma(s) = (\ddot{x}(s), \ddot{y}(s))$. Hence, the curvature is $\kappa_\gamma(s) = \sqrt{\ddot{x}^2(s) + \ddot{y}^2(s)}$. The defining condition of the clothoid curve is that the derivative of $\kappa_\gamma(s)$ is constant, or, equivalently, that

$$\ddot{\kappa}_\gamma(s) = 0 \Leftrightarrow \frac{d^2}{ds^2}\left(\sqrt{\ddot{x}^2(s) + \ddot{y}^2(s)}\right) = 0$$

(b) Let $\theta(s)$ be the angle of the unit tangent $T_\gamma(s) = (\dot{x}(s), \dot{y}(s))$ with the $x$ axis; this means (since $T_\gamma$ is unit) that $\cos(\theta(s)) = \dot{x}(s)$ and $\sin(\theta(s)) = \dot{y}(s)$. Then,

$$\kappa_\gamma(s) = \sqrt{\ddot{x}^2(s) + \ddot{y}^2(s)} = \sqrt{\sin^2(\theta)\dot{\theta}^2 + \cos^2(\theta)\dot{\theta}^2} = |\dot{\theta}(s)|.$$

Thus, the condition derived before implies that

$$\frac{d^2}{ds^2}|\dot{\theta}(s)| = 0.$$

Initially, at $s = 0$, our initial conditions imply that $\theta(0) = 0$ and $\dot{\theta}(s) = 0$. Therefore, integrating the above, we get that $|\dot{\theta}(s)| = \lambda s$ for some non-zero constant $\lambda > 0$. Assuming, without loss of generality, that $\dot{\theta}(s) > 0$ for $s > 0$, integrating this relation once more we obtain

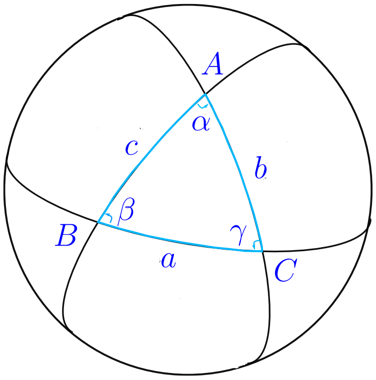
$$\theta(s) = \frac{1}{2}\lambda s^2.$$

Thus,

$$(\dot{x}(s), \dot{y}(s)) = (\cos(\theta(s)), \sin(\theta(s))) = \left(\cos\left(\frac{1}{2}\lambda s^2\right), \sin\left(\frac{1}{2}\lambda s^2\right)\right).$$

Upon integration (using that $(x(0), y(0)) = (0, 0)$), we finally obtain that

$$x(s) = \int_0^s \cos(\frac{1}{2}\lambda\sigma^2)\, d\sigma = \sqrt{\frac{2}{\lambda}}\int_0^{\sqrt{\frac{\lambda}{2}}s} \cos(\tau^2)\, d\tau, \quad y(s) = \sqrt{\frac{2}{\lambda}}\int_0^{\sqrt{\frac{\lambda}{2}}s} \sin(\tau^2)\, d\tau.$$



A *spherical triangle* is defined by three points $A, B, C$ on a sphere $\mathbb{S}$, with the great-circle arcs $a$ (joining $B$ and $C$), $b$ (joining $A$ and $C$), and $c$ (joining $A$ and $B$). These arcs are the sides of the spherical triangle. We denote by $\alpha$ the angle formed at $A$ by arcs $b$ and $c$, similarly $\beta$ at $B$ and $\gamma$ at $C$.

Recall: A great circle on a sphere is the circle obtained by intersecting the sphere with a plane passing through the center of the sphere. Other circles are called small circles. Two points on a sphere are always connected by two great-circle arcs; in defining a spherical triangle, we only consider the shorter one.

**4.7** By abuse of notation, we also denote by $a$, $b$, and $c$ the lengths of the sides of the spherical triangle. Prove the following spherical trigonometry formula:

$$\cos\left(\tfrac{c}{r}\right) = \cos\left(\tfrac{a}{r}\right)\cos\left(\tfrac{b}{r}\right) + \sin\left(\tfrac{a}{r}\right)\sin\left(\tfrac{b}{r}\right)\cos(\gamma),$$

where $r$ is the radius of the sphere.

**Solution.** We present two proofs.

**(1) Vector (Lagrange identity) proof.** Take $r = 1$ (unit sphere centered at the origin). Represent $A, B, C$ as unit vectors in $\mathbb{R}^3$. The angle $\gamma$ is the angle between the planes $OAC$ and $OBC$, i.e. between normals $C \times A$ and $C \times B$. So

$$\cos\gamma = \frac{\langle C \times A,\, C \times B\rangle}{\|C \times A\|\,\|C \times B\|}.$$

By Lagrange's identity,

$$\langle C \times A,\, C \times B\rangle = \langle C, C\rangle\langle A, B\rangle - \langle C, B\rangle\langle C, A\rangle = \cos c - \cos a \cos b.$$

Also $\|C \times A\| = \sin b$, $\|C \times B\| = \sin a$. Hence

$$\cos\gamma = \frac{\cos c - \cos a \cos b}{\sin a \sin b},$$

which rearranges to the stated spherical cosine law. For general radius $r$ scale arc lengths by $1/r$.

**(2) Geometric vector decomposition.** Let $U$ be the unit vector in plane $OAC$ orthogonal to $C$ such that $A = \cos b\, C + \sin b\, U$, and let $V$ be the unit vector in plane $OBC$ orthogonal to $C$ with $B = \cos a\, C + \sin a\, V$. Then

$$\cos c = \langle A, B\rangle = \cos a \cos b + \sin a \sin b \langle U, V\rangle = \cos a \cos b + \sin a \sin b \cos \gamma.$$

**4.8** The spherical distance $d_{\mathbb{S}}(A, B)$ between two points $A$ and $B$ on a sphere $\mathbb{S}$ is by definition the length of the great-circle arc joining these points. Show, using spherical trigonometry, that $(\mathbb{S}, d_{\mathbb{S}})$ satisfies all the properties of a metric space.

**Solution.** The nontrivial property to show is the triangle inequality:

$$d_{\mathbb{S}}(A, B) \leqslant d_{\mathbb{S}}(A, C) + d_{\mathbb{S}}(C, B).$$

Let the corresponding arc lengths be $a, b, c$ on the unit sphere. By the spherical cosine law,

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$$

Since $a, b \in [0, \pi]$ we have $\sin a, \sin b \geqslant 0$, and hence

$$\cos c \geqslant \cos a \cos b - \sin a \sin b = \cos(a + b).$$

On $[0, \pi]$ the cosine function is decreasing, thus $\cos c \geqslant \cos(a + b)$ implies $c \leqslant a + b$, i.e. the triangle inequality. Equality occurs only when $\gamma = \pi$ (degenerate spherical triangle).

**Bonus exercise (not part of the examinable course material).**

**4.9** The goal of this exercise is to show that one can (re)define the length of a $C^1$ curve by a process of “polygonal approximations.” Let $\gamma : [a, b] \to \mathbb{R}^n$ be a $C^1$ curve, and let $\sigma = [t_0 = a < t_1 < \cdots < t_m = b]$ be a subdivision of $[a, b]$. We set

$$L(\gamma) = \sup_{\sigma} \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all subdivisions of $[a, b]$ and $d(p, q) = \|q - p\|$.

**(a)** Make a drawing and briefly explain the meaning of this formula.

**(b)** Show that for every $C^1$ curve we have $L(\gamma) \leqslant \ell(\gamma)$, where $\ell(\gamma)$ is the length of $\gamma$ as defined in the course.

**(c)** Prove the reverse inequality $\ell(\gamma) \leqslant L(\gamma)$. (Hint: use that $\dot\gamma$ is uniformly continuous and show that for every $\varepsilon > 0$ one can find a sufficiently fine subdivision of $[a, b]$ such that

$$\ell(\gamma) \leqslant \sum_{i=1}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) + 2\varepsilon(b-a).$$

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**Solution.**

(a) The idea is to approximate the curve by a polygonal path joining successive sample points $\gamma(t_i)$. The length of the polygonal approximation is the sum of the Euclidean distances between consecutive vertices; taking the supremum over all subdivisions recovers the intuitive length of the curve.

(b) For any $a \leqslant t' < t'' \leqslant b$, the distance between endpoints satisfies

$$d(\gamma(t'), \gamma(t'')) \leqslant \int_{t'}^{t''} V_\gamma(t)\, dt,$$

where $V_\gamma = \|\dot\gamma\|$. Summing over the subdivision and integrating yields

$$\sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) \leqslant \int_a^b V_\gamma(t)\, dt = \ell(\gamma).$$

Taking the supremum over all subdivisions gives $L(\gamma) \leqslant \ell(\gamma)$.

(c) Since $\dot\gamma$ is uniformly continuous on $[a, b]$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\dot\gamma(s) - \dot\gamma(t)| < \varepsilon$ whenever $|s-t| < \delta$. Choose a subdivision $\sigma$ with mesh $\max_i(t_{i+1} - t_i) < \delta$. For each interval $[t_i, t_{i+1}]$ and any $t \in (t_i, t_{i+1}]$ we have $\|\dot\gamma(t)\| \leqslant \|\dot\gamma(t_i)\| + \varepsilon$. Integrating,

$$\int_{t_i}^{t_{i+1}} \|\dot\gamma(t)\|\, dt \leqslant \|\dot\gamma(t_i)\|(t_{i+1} - t_i) + \varepsilon(t_{i+1} - t_i).$$

But

$$\|\dot\gamma(t_i)\|(t_{i+1} - t_i) = \Big\| \int_{t_i}^{t_{i+1}} \dot\gamma(t_i)\, dt \Big\| \leqslant \Big\| \int_{t_i}^{t_{i+1}} \dot\gamma(t)\, dt \Big\| + \Big\| \int_{t_i}^{t_{i+1}} (\dot\gamma(t_i) - \dot\gamma(t))\, dt \Big\|.$$

Estimating the latter term by $\varepsilon(t_{i+1} - t_i)$ and using $\int_{t_i}^{t_{i+1}} \dot\gamma(t)\, dt = \gamma(t_{i+1}) - \gamma(t_i)$, we get

$$\int_{t_i}^{t_{i+1}} \|\dot\gamma(t)\|\, dt \leqslant \|\gamma(t_{i+1}) - \gamma(t_i)\| + 2\varepsilon(t_{i+1} - t_i).$$

Summing over $i$ yields

$$\ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\, dt \leqslant \sum_{i=0}^{m-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| + 2\varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary and the middle sum is bounded by $L(\gamma)$, we conclude $\ell(\gamma) \leqslant L(\gamma)$. Combined with (b), this gives equality $\ell(\gamma) = L(\gamma)$ for $C^1$ curves.

**Remark on curve length.** The previous exercises show that for a $C^1$ curve $\gamma : [a,b] \to \mathbb{R}^n$,

$$\sup_{\sigma} \sum_{i=0}^{m-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| = \int_a^b \|\dot{\gamma}(t)\|\, dt.$$

This equality remains valid for piecewise $C^1$ curves. More generally, a curve is rectifiable ($L(\gamma) < \infty$) iff each coordinate function has bounded variation; moreover $\ell(\gamma) = L(\gamma) < \infty$ iff each coordinate function is absolutely continuous. There exist rectifiable curves with $\ell(\gamma) < L(\gamma)$ in greater generality (example: graph of Cantor–Vitali function).